

Braid Groups

Notes on Artin presentations,
configuration spaces, and mapping classes

Ian X

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Abstract

A six-chapter development of the braid group for $n \geq 3$: the Artin presentation, loops in unordered configuration spaces, and boundary-fixed mapping classes of a punctured disk. Definitions, prerequisites, proofs, examples, and the two comparison isomorphisms are collected from the Get Curious article series.

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1 Scope and structure

Chapter dependencies. Chapters 2 and 3 feed chapter 5; chapters 3 and 4 feed chapter 6.

A braid can be described by a word, by a collision-free motion of points, or by a deformation class of a punctured disk. This series defines each model and proves that they give isomorphic groups. Throughout, $n > 2$, so $n \geq 3$. We leave the smaller cases outside the scope.

1.1 Three models of the braid group

Write $D = \{z \in \mathbb{C} : |z| \leq 1\}$, let D° be its interior, and fix the unordered set $Q = \{q_1, \dots, q_n\}$, where $q_j = -1 + 2j/(n+1)$.

Definition 1.1 (Artin presentation). The Artin group A_n has formal generators $s_1, \dots, s_{n-1}$ and relations

$$s_i s_j = s_j s_i \quad (|i - j| \geq 2), \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (1 \leq i \leq n-2).$$

Definition and relation ranges.

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Definition 1.2 (Configuration-space braid group). Let

$$F_n(D^\circ) = \{(z_1, \dots, z_n) \in (D^\circ)^n : z_i \neq z_j \text{ for } i \neq j\}, \quad C_n(D^\circ) = F_n(D^\circ)/S_n.$$

The group $\pi_1(C_n(D^\circ), Q)$ consists of based loops of *unordered* configurations, modulo endpoint-fixed homotopy. Particles remain distinct, but may return in permuted positions. Configuration-space definition.

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Definition 1.3 (Boundary-fixed mapping class group). The group $\text{Mod}(D, Q; \partial D)$ consists of isotopy classes of orientation-preserving homeomorphisms of D which fix ∂D pointwise and preserve Q setwise. The isotopies obey these same conditions at every time. Mapping-class definition.

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We use s_i for abstract generators and σ_i for their elementary configuration-loop counterparts, defined in chapter 3.

Theorem 1.4 (Comparison of the three models).

$$A_n \cong \pi_1(C_n(D^\circ), Q) \cong \text{Mod}(D, Q; \partial D).$$

For every integer $n \geq 3$, the displayed groups are isomorphic.

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Proof. The homomorphism $s_i \mapsto [\sigma_i]$ and its crossing-word inverse are constructed in Chapter 5. The inverse-endpoint homomorphism to mapping classes and its bijectivity are proved in Chapter 6. □

After these isomorphisms are established, we use B_n for the braid group in any of the three models. Ordered configurations and homeomorphisms fixing each marked point describe the corresponding *pure* subgroups.

1.2 Structure of the series

1. **This chapter:** scope, conventions, and dependencies.
2. The Artin presentation: words, free groups, normal closures, quotient groups, and presentations.
3. Configuration spaces: paths, homotopies, fundamental groups, unordered configurations, and lifted strands.
4. Mapping classes: homeomorphisms, boundary-fixed isotopies, and half twists.
5. The first equivalence: show that elementary exchanges satisfy the Artin relations, generate every loop, and have no additional relations.
6. The second equivalence: lift point motions to disk homeomorphisms and use the Alexander contraction to prove the resulting map is an isomorphism.

1.3 Prerequisites and conventions

We assume basic topology, metric spaces, Euclidean spaces, and elementary group operations. We develop free groups and presentations in chapter 2, the required path and homotopy theory in chapter 3, and isotopies in chapter 4. Further algebraic-topology and general-position tools appear in collapsible sections where they are used.

For loops, $\alpha * \beta$ means “traverse α , then β .” Mapping classes use ordinary composition, $[f][g] = [f \circ g]$. Consequently chapter 6 sends a lifted motion F_t , starting at the identity, to $[F_1^{-1}]$. An elementary positive configuration loop exchanges two adjacent points counter-clockwise; its mapping class under this convention is a clockwise half twist.

The open disk and plane configuration models are interchangeable by an explicit homeomorphism, explained in chapter 3. The boundary condition in the mapping-class model remains pointwise fixation.

1.4 Sources

Direct definitions are linked above. For the equivalences, see Alphabeta Math’s comparison of braid models and Juan González-Meneses, *Basic results on braid groups*, §§1.2–1.4. The later chapters supply the proofs.

2 The Artin presentation

A group presentation turns symbols and equations into a group. This chapter constructs that process before applying it to braids. Fix an integer $n \geq 3$. Here A_n means the abstract group defined by the Artin presentation; its identification with moving points is the theorem proved in Chapter 5. After the equivalences are established, B_n is the common name for the braid group in any of its three models.

Prerequisites: groups, homomorphisms, and subgroups

A *group* is a set with an associative multiplication, an identity 1, and an inverse for each element. A *homomorphism* $\phi : G \rightarrow H$ satisfies $\phi(gh) = \phi(g)\phi(h)$. A subgroup is a subset that contains 1 and is closed under products and inverses. The subgroup generated by a subset consists of all finite products of its elements and their inverses, including the empty product 1.

The kernel $\ker \phi = \{g : \phi(g) = 1\}$ is a subgroup and satisfies $g(\ker \phi)g^{-1} = \ker \phi$: apply ϕ to gkg^{-1} . A subgroup with this conjugation property is called *normal*. See Alphabeta Math's groups and subgroups and normal subgroups and quotients.

2.1 Words and reduction

Free groups retain exactly the cancellations forced by inverses. Additional equations will be imposed only after this construction.

Definition 2.1 (Words and elementary reduction). Let S be a set of symbols. Make a disjoint set of formal inverse symbols S^{-1} , with $(s^{-1})^{-1} = s$. A *word* is a finite string of letters in $S \sqcup S^{-1}$. The empty word is denoted ε ; concatenation puts one word after another. If $w = \ell_1 \cdots \ell_k$, its inverse word is $w^{-1} = \ell_k^{-1} \cdots \ell_1^{-1}$.

An elementary cancellation deletes an adjacent pair $\ell\ell^{-1}$. A word is *reduced* if no such pair occurs. Define $u \sim v$ when finitely many cancellations or reverse insertions transform u into v . These moves may occur inside any surrounding words, so $u \sim v$ implies $aub \sim avb$.

Alphabeta Math

Theorem 2.2 (Reduced normal form). *Every word is equivalent to a unique reduced word. Any sequence of elementary cancellations terminates at that word.*

Alphabeta Math

Proof. Scan a word from left to right, keeping a stack of letters. When the next letter is inverse to the stack's last letter, remove that last letter; otherwise append the new letter. Write $R(w)$ for the final stack. It is reduced, and each scan step is either concatenation or one allowed cancellation, so $R(w) \sim w$.

For a letter ℓ , let T_ℓ be the operation of processing ℓ on a reduced stack. Its inverse is $T_{\ell^{-1}}$. If ℓ was appended, the next operation removes it. If ℓ removed a terminal ℓ^{-1} , the next operation restores that letter: the remaining stack cannot end in ℓ , because the original stack was reduced. Thus processing an inserted pair $\ell\ell^{-1}$ has no effect, even after a nonempty prefix. Processing the remaining suffix then gives the same result. Hence R is unchanged by every allowed move.

Every reduced word r satisfies $R(r) = r$. Consequently, if two reduced words are equivalent, both equal the same $R(w)$. Repeated cancellation terminates because length decreases, and its endpoint must be this unique reduced word. Compare the reduced-word theorem in Alphabeta Math.

□

Word reduction. Only adjacent inverse letters cancel. A reduced word can still become trivial after extra relations are imposed. Reduction here takes place in the free group.

2.2 Free groups

Definition 2.3 (The free group on a set). Let S be a set. The free group $F(S)$ is the set of word classes under elementary insertion and cancellation, with multiplication $[u][v] = [uv]$, identity $[\varepsilon]$, and inverse $[w]^{-1} = [w^{-1}]$.

Alphabeta Math

Proposition 2.4 (Group structure). *The operations in Definition 2.3 are well defined and make $F(S)$ a group. The map $S \rightarrow F(S)$, $s \mapsto [s]$, is injective.*

Alphabeta Math: group structure · generator injection

Proof. Compatibility with surrounding words makes this well-defined. Concatenation is associative, so this multiplication is associative. The identity is $[\varepsilon]$. Successively cancelling from the middle of ww^{-1} and $w^{-1}w$ shows that $[w^{-1}]$ is the inverse of $[w]$. Thus $F(S)$ is a group, and its elements may be represented uniquely by reduced words.

The symbols $s \in S$ give distinct elements $[s]$, since distinct one-letter words are reduced. We identify each symbol with its class when no ambiguity results. This group is *free on S* in the following precise sense:

□

Theorem 2.5 (Universal property of the free group). *For every group H and every function $f : S \rightarrow H$, there is exactly one homomorphism $\hat{f} : F(S) \rightarrow H$ extending f .*

Alphabeta Math

Proof.

$$\hat{f}([s_1^{\epsilon_1} \cdots s_k^{\epsilon_k}]) = f(s_1)^{\epsilon_1} \cdots f(s_k)^{\epsilon_k}, \quad \epsilon_j \in \{1, -1\}.$$

A cancellation or insertion evaluates to multiplying by $f(s)f(s)^{-1} = 1$, so the formula is well-defined. Concatenation evaluates to multiplication, which proves the homomorphism property. Every word is a product of generator classes and their inverses, so any extending homomorphism must have this formula. This also proves uniqueness. “Free” therefore means that assigning the generators requires no compatibility conditions beyond those already built into a group.

□

2.3 Normal closures and quotients

Suppose we want words u and v to become equal. In a group this means making $uv^{-1} = 1$. Such a word is called a *relator*. If a relator becomes the identity, its inverse and every conjugate wrw^{-1} must also become the identity, as must products of those elements. Killing just a subgroup generated by relators would not generally account for conjugation.

Definition 2.6 (Normal closure). For $\mathcal{R} \subseteq F(S)$, its normal closure $\langle\langle \mathcal{R} \rangle\rangle$ is the smallest normal subgroup of $F(S)$ containing $\mathcal{R}$.

Alphabeta Math

Lemma 2.7 (Products of conjugates). *The normal closure of $\mathcal{R}$ consists exactly of the empty product and the products*

$$(w_1 r_1^{\epsilon_1} w_1^{-1}) \cdots (w_k r_k^{\epsilon_k} w_k^{-1}), \quad r_j \in \mathcal{R}, \quad w_j \in F(S), \quad \epsilon_j \in \{1, -1\}.$$

Alphabeta Math

Proof. Include the empty product. Products stay in this set, inverses reverse the factors and change their signs, and conjugating a factor by g replaces w_j by gw_j . The set is therefore a normal subgroup containing $\mathcal{R}$. Every normal subgroup containing $\mathcal{R}$ contains all these factors and products, proving minimality. See also normal closure as products of conjugates. $\square$

Definition 2.8 (Cosets and the quotient set). If $N \trianglelefteq G$, define $gN = \{gn : n \in N\}$ and $G/N = \{gN : g \in G\}$. Two representatives define the same coset precisely when their quotient lies in N .

Alphabeta Math: cosets $\cdot$ quotients

Proposition 2.9 (Quotient group). *For $N \trianglelefteq G$, the rule below makes G/N a group. The projection $q(g) = gN$ is a surjective homomorphism with kernel N .*

$$(gN)(hN) = ghN.$$

Alphabeta Math

Proof. Cosets satisfy $gN = hN$ if and only if $h = gn$ for some $n \in N$. If the cosets are equal, $h \in gN$; conversely, $h = gn$ gives $hN = gnN = gN$. Thus the coset equality criterion used in the definition follows from the subgroup laws.

If $g' = gn$ and $h' = hm$, then $g'h' = gh(h^{-1}nh)m$. Normality puts $h^{-1}nh$ in N , so the product does not depend on the representatives. Associativity, identity N , and inverse $g^{-1}N$ now follow from the corresponding laws in G . The projection $q : G \rightarrow G/N$, $q(g) = gN$, is a surjective homomorphism with kernel N . $\square$

Proposition 2.10 (Universal property of the quotient). *If $\phi : G \rightarrow H$ is a homomorphism and $N \subseteq \ker \phi$, there is a unique homomorphism $\bar{\phi} : G/N \rightarrow H$ satisfying $\bar{\phi} \circ q = \phi$.*

Alphabeta Math

Proof. Moreover, if $\phi : G \rightarrow H$ kills N , there is exactly one homomorphism $\bar{\phi} : G/N \rightarrow H$ with $\bar{\phi}(gN) = \phi(g)$. This is well-defined because $\phi(gn) = \phi(g)$; the formula also proves uniqueness and multiplicativity. This is the quotient's universal property. $\square$

2.4 Group presentations

Imposing a relator also imposes its conjugates and products. The normal closure records exactly these consequences.

Definition 2.11 (Group presentation). A *presentation* defines the group

$$\langle S \mid \mathcal{R} \rangle := F(S) / \langle\langle \mathcal{R} \rangle\rangle.$$

Writing an equation $u = v$ in the relation list means including the relator uv^{-1} . Two words represent the same element precisely when $uv^{-1} \in \langle\langle \mathcal{R} \rangle\rangle$: indeed $uN = vN$ is equivalent to $v^{-1}u \in N$, and normality makes this equivalent to $uv^{-1} \in N$. See Alphabeta Math's definition of a group presentation.

Alphabeta Math

Proposition: equality as a finite sequence of relation moves

Proposition 2.12 (Relation moves). *Two words define the same element of a presented group if and only if free insertions, cancellations, and replacements by defining equations connect them.*

Alphabeta Math: normal closure · presentations

Proof. Besides free insertion and cancellation, allow replacing either side of a defining equation by the other inside any word. Every move preserves its quotient element. Conversely, if a word is a product of conjugates of relators and their inverses, choose word representatives for its factors. The original word and this expanded product have the same reduced word, so free insertions and cancellations connect them. A relator uv^{-1} becomes vv^{-1} by replacing u with v , then cancels; its inverse vu^{-1} cancels in the same way. Each conjugate therefore becomes the empty word. The normal-closure criterion proves that every word equal to the identity can be transformed to the empty word by finitely many such moves. Applying this to uv^{-1} , and appending v , proves the corresponding statement for equality of any two words. Insertions and reverse relation moves are essential; a delete-only procedure need not succeed. $\square$

Theorem 2.13 (Homomorphisms from a presentation). *An assignment $S \rightarrow H$ extends uniquely to a homomorphism $\langle S \mid \mathcal{R} \rangle \rightarrow H$ if and only if every relator evaluates to the identity.*

Alphabeta Math

Proof. To define a homomorphism from $\langle S \mid \mathcal{R} \rangle$ to H , assign each generator an element of H and check that every relator evaluates to 1. The free-group property gives a homomorphism on $F(S)$. Its kernel is normal and contains $\mathcal{R}$, hence contains the normal closure; the quotient property then gives the desired homomorphism. Conversely, any homomorphism out of the presented group must respect those equations. $\square$

Example 2.14 (A cyclic presentation). For $m \geq 1$, $\langle a \mid a^m = 1 \rangle \cong \mathbb{Z}/m\mathbb{Z}$.

Alphabeta Math

Proof. A reduced word on one generator is a^k , and multiplication adds exponents. The normal closure of a^m is the subgroup of powers with exponent divisible by m , so quotienting identifies exponents modulo m . This illustrates why a presentation asserts both the listed equations and exactly their group-theoretic consequences. $\square$

2.5 The Artin braid group

Definition 2.15 (Artin braid group). Take $S = \{s_1, \dots, s_{n-1}\}$. The abstract *Artin group* A_n is

$$A_n = \left\langle s_1, \dots, s_{n-1} \mid \begin{array}{ll} s_i s_j = s_j s_i & (1 \leq i, j \leq n-1, |i-j| \geq 2), \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & (1 \leq i \leq n-2) \end{array} \right\rangle.$$

These are the *distant commutation* and *adjacent braid* relations. This is a definition at this stage, matching the Artin presentation entry. For a standard research survey of the algebraic and geometric definitions, see Birman and Brendle, *Braids: A Survey*, §1.2, equation (2).

Alphabeta Math

Remark 2.16 (Relators and notation). For complete precision, the relator set contains

$$s_i s_j s_i^{-1} s_j^{-1} \quad (|i - j| \geq 2),$$

and

$$s_i s_{i+1} s_i s_{i+1}^{-1} s_i^{-1} s_{i+1}^{-1} \quad (1 \leq i \leq n - 2).$$

The symbols s_i first denote free generators and then their images in the quotient; we reserve σ_i for the configuration loops in chapter 3. For $n = 3$, there are two generators and just the adjacent relation $s_1 s_2 s_1 = s_2 s_1 s_2$. For $n = 4$, the new distant relation is $s_1 s_3 = s_3 s_1$.

Both sides of an adjacent braid relation are distinct reduced free words. The quotient makes them equal. Thus free reduction alone does not give a unique representative for a braid. A presentation provides a criterion for equality through normal closure, but it does not by itself supply an efficient equality algorithm.

Alphabeta Math

2.6 Permutations and exponent sum

Proposition 2.17 (Two quotient homomorphisms). *There are surjective homomorphisms $p : A_n \rightarrow S_n$ and $e : A_n \rightarrow \mathbb{Z}$ determined by $p(s_i) = (i \ i + 1)$ and $e(s_i) = 1$. Each s_i has infinite order.*

Alphabeta Math: permutation map · exponent sum

Proof. Permutation map. Let S_n be the group of permutations of $\{1, \dots, n\}$, with ordinary function composition. Set $t_i = (i \ i + 1)$. Disjoint transpositions commute. On the three affected entries, both $t_i t_{i+1} t_i$ and $t_{i+1} t_i t_{i+1}$ exchange i and $i + 2$ and fix $i + 1$; they fix every other entry. The presentation property therefore gives a homomorphism

$$p : A_n \longrightarrow S_n, \quad p(s_i) = t_i.$$

It is surjective: repeatedly swapping adjacent out-of-order entries sorts any list of distinct numbers, so adjacent transpositions generate every permutation. The equations $t_i^2 = 1$ in the target are extra equations, not part of the defining braid relations. This algebraic map uses usual composition; a chronological record of labelled trajectories requires an explicit convention, as discussed in Chapter 3.

Exponent sum. Assign every s_i the integer 1 in the additive group $\mathbb{Z}$. Both sides of each relation have the same sum, so there is a homomorphism

$$e : A_n \longrightarrow \mathbb{Z}, \quad e(s_{i_1}^{\epsilon_1} \cdots s_{i_k}^{\epsilon_k}) = \sum_{j=1}^k \epsilon_j.$$

It follows that every s_i has infinite order: if $s_i^m = 1$, applying e gives $m = 0$. In particular $s_i^2 \neq 1$, although $p(s_i^2) = 1$. Exponent sum is only a partial test: $s_1 s_2^{-1}$ has sum zero but is nontrivial, because its image $t_1 t_2$ is nonidentity.

Surjectivity of e follows from $e(s_1) = 1$.

□

2.7 Comparison with geometric braids

Eventually the formal generator s_i will correspond to the configuration-loop class $[\sigma_i]$, a counterclockwise exchange of the i -th and $(i + 1)$ -st points in a real-axis base configuration: the left point travels below and the right point above. That interpretation is a candidate

correspondence, not a consequence of the symbols alone. Showing that geometric exchanges satisfy the relations constructs a homomorphism; showing that there are no additional geometric relations proves that it is an isomorphism. Chapter 5 addresses both tasks after Chapter 3 defines configuration-space braids. Chapter 4 introduces the third model, and Chapter 6 compares it with moving points.

2.8 Sources and further reading

- Joan S. Birman and Tara E. Brendle, *Braids: A Survey*, §§1.1–1.3: the classical definitions; §1.2, equation (2): the Artin presentation.
- Alfabeta Math: reduced words form the free group, normal closure as products of conjugates, and group presentations. The proofs needed here are given above.
- Alfabeta Math: the braid group by Artin presentation.

3 Configuration spaces

1. Paths and endpoint-fixed homotopy $\rightarrow$
2. Loops form the fundamental group $\rightarrow$
3. Distinct points form configuration spaces $\rightarrow$
4. Lift a loop to labelled trajectories $\rightarrow$
5. Braids, pure braids, and endpoint permutations

A braid records the motion of distinct points. The right space therefore has a point for every possible configuration, and a braid becomes a loop in that space. Deforming the braid becomes deforming the loop. We build this definition from paths, then explain exactly why forgetting labels produces the full braid group.

3.1 Paths and the fundamental group

Definition 3.1 (Paths and based loops). Write $I = [0, 1]$. A **path** in a topological space X is a continuous map $\alpha : I \rightarrow X$. Its endpoints are $\alpha(0)$ and $\alpha(1)$. A **loop based at** x_0 has both endpoints equal to x_0 .

Alphabeta Math

Definition 3.2 (Homotopy relative to a subspace). A **homotopy** from a continuous map $f : Y \rightarrow X$ to $g : Y \rightarrow X$ is a continuous $H : Y \times I \rightarrow X$ with $H(y, 0) = f(y)$ and $H(y, 1) = g(y)$. It is **relative to** $A \subseteq Y$ when $H(a, s) = f(a) = g(a)$ for every $a \in A$. Path homotopy is the case $Y = I$, $A = \{0, 1\}$.

Two paths α, β with the same endpoints are **homotopic with endpoints fixed** if there is a continuous map $H : I \times I \rightarrow X$ such that

$$H(t, 0) = \alpha(t), \quad H(t, 1) = \beta(t), \quad H(0, s) = \alpha(0), \quad H(1, s) = \alpha(1).$$

Alphabeta Math

The parameter t runs along a path; s runs through the deformation. For based loops the entire left and right edges of this square stay at x_0 . Write $[\alpha]$ for the endpoint-fixed homotopy class. Fixing endpoints matters: it keeps the starting configuration of a braid unchanged throughout a deformation.

Definition 3.3 (Fundamental group). The **fundamental group** $\pi_1(X, x_0)$ is the set of these based-loop classes. Multiply by following the first loop and then the second:

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2}, \\ \beta(2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases} \quad [\alpha][\beta] = [\alpha * \beta].$$

Alphabeta Math

Prerequisite: pasting and Lebesgue numbers

Lemma 3.A (Compactness tools). *A finite closed cover permits pasting continuous maps agreeing on overlaps. Every open cover of a compact metric space has a positive Lebesgue number.*

Alphabeta Math

Proof. The **finite closed pasting lemma** says that continuous maps on finitely many closed subsets covering a space combine to a continuous map if they agree on overlaps. For a closed set in the target, its inverse image is a finite union of sets closed in those closed subsets, hence closed in the whole domain. This proves the lemma and applies to interval subdivisions and rectangular grids.

The **Lebesgue-number lemma** says that an open cover of a compact metric space has a number $\delta > 0$ such that every subset of diameter less than δ lies in one cover member. To prove it, choose for each point x a radius $r_x > 0$ with the ball $B(x, 2r_x)$ contained in a cover member. Finitely many balls $B(x, r_x)$ cover the space. Take δ to be the minimum of those finitely many radii. A nonempty set of diameter less than δ , containing a point of one of these smaller balls, lies in the corresponding larger ball by the triangle inequality. Thus sufficiently short interval pieces or sufficiently small grid cells fit into single cover members. $\square$

Proposition 3.4 (The fundamental-group laws). *Endpoint-fixed homotopy is an equivalence relation. Concatenation induces a group law on based-loop classes, with constant-loop identity and reversed-path inverse.*

Alphabeta Math

Proof of Proposition 3.4

Proof. The two pieces agree at the join, so the pasting lemma makes this path continuous. The identity is the constant loop $c_{x_0}(t) = x_0$, and the inverse is $\bar{\alpha}(t) = \alpha(1 - t)$. These operations satisfy the group laws on homotopy classes; the justification is below. See also Alphabeta Math's definition and group-law proof.

Endpoint-fixed homotopy is an equivalence relation: the constant deformation proves reflexivity, replacing s by $1 - s$ proves symmetry, and concatenating two deformations in their s -coordinate proves transitivity. The pasting lemma applies because they agree along the seam. Concatenating two endpoint-fixed homotopies in their t -coordinate shows that the product does not depend on representatives.

Changing speed does not change a path class. If $r : I \rightarrow I$ is continuous with $r(0) = 0, r(1) = 1$, then $\alpha((1-s)t + sr(t))$ deforms α into $\alpha \circ r$, fixing endpoints. The two bracketings of three loops have the same successive segments but different time allocations. Interpolating their piecewise-linear time schedules gives their homotopy, proving associativity. The same argument removes the pause in $c * \alpha$ or $\alpha * c$, proving the identity laws.

For cancellation set $f(t) = 2t$ for $t \leq 1/2$ and $f(t) = 2 - 2t$ for $t \geq 1/2$. The formula $H(t, s) = \alpha((1-s)f(t))$ contracts $\alpha * \bar{\alpha}$ to the constant loop at $\alpha(0)$, keeping endpoints fixed. Apply the same formula to $\bar{\alpha}$ for the other inverse law. These explicit deformations prove all group axioms. $\square$

Prerequisite: induced maps and change of basepoint

Proposition 3.B (Functoriality and basepoint transport). *A based continuous map induces a homomorphism of fundamental groups. A path $u : x_0 \rightarrow x_1$ induces an isomorphism $[\alpha] \mapsto [\bar{u} * \alpha * u]$.*

Alphabeta Math: induced maps · basepoint transport

Proof. A continuous map $f : X \rightarrow Y$ with $f(x_0) = y_0$ induces $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$, $[\alpha] \mapsto [f \circ \alpha]$. Composing homotopies with f proves well-definedness, and composing concatenated loops proves that it is a homomorphism. A homeomorphism gives an isomorphism, since its inverse induces the inverse map.

If u is a path from x_0 to x_1 , the map $[\alpha] \mapsto [\bar{u} * \alpha * u]$ is an isomorphism from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$. Adjacent $u * \bar{u}$ segments cancel by the contraction above; reversing u gives the inverse. A different choice of connecting path may change this isomorphism by conjugation. Thus a path-connected space has isomorphic fundamental groups at different basepoints, but no preferred identification without a choice of path. Alphabeta Math proves this basepoint-change statement.

Only products, subspaces, quotient topology, and the lifting argument below are needed here. Higher homotopy groups, covering classification, and homology are not prerequisites for the definition. □

3.2 Ordered and unordered configurations

Fix $n \geq 3$. Let $D = \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disk and $D^\circ = \{z : |z| < 1\}$ its interior. Choose the ordered base configuration $q = (q_1, \dots, q_n)$, with $q_j = -1 + 2j/(n+1)$, and its underlying set $Q = \{q_1, \dots, q_n\}$.

Definition 3.5 (Configuration spaces). For a space X , its **ordered configuration space** is

$$F_n(X) = \{(z_1, \dots, z_n) \in X^n : z_i \neq z_j \text{ whenever } i \neq j\},$$

Give $F_n(X)$ the subspace topology of X^n . Let S_n act by

$$(\tau \cdot z)_j = z_{\tau^{-1}(j)}.$$

The unordered configuration space is the orbit space with its quotient topology:

$$C_n(X) = F_n(X)/S_n.$$

An orbit is precisely an n -element subset of X . Let $p : F_n(X) \rightarrow C_n(X)$ forget the ordering. The **quotient topology** means that $V \subset C_n(X)$ is open exactly when $p^{-1}(V)$ is open in $F_n(X)$. This topology, rather than merely the set of subsets, makes continuity of unordered motion meaningful. These are the ordered and unordered definitions used by Alphabeta Math.

Alphabeta Math: ordered configurations $\cdot$ unordered configurations

Lemma 3.6 (Free permutation action). *For $X = D^\circ$, the space $F_n(X)$ is open in $\mathbb{R}^{2n}$, and the displayed permutation action is a free action by homeomorphisms.*

Alphabeta Math

Proof. A coordinate is a label: the first point remains point 1 as it moves. When $X = D^\circ$, this is an open subset of $\mathbb{R}^{2n}$: at any tuple, the positive minimum separation and positive distances from the boundary allow sufficiently small independent movements without collisions or escape.

The symmetric group S_n , the group of permutations of $\{1, \dots, n\}$, acts by

Coordinate permutations are homeomorphisms, and this is a left action because $(\tau\nu)^{-1} = \nu^{-1}\tau^{-1}$. It is free: if $\tau \cdot z = z$, distinct coordinates force $\tau(j) = j$ for every j . □

Definition 3.7 (Covering map). A **covering map** is a map for which every point downstairs has an open neighbourhood whose inverse image is a disjoint union of open sets, each mapped homeomorphically onto that neighbourhood.

Alphabeta Math

Theorem 3.8 (The configuration covering). *The forgetful map $p : F_n(D^\circ) \rightarrow C_n(D^\circ)$ is an $n!$ -sheeted covering.*

Alphabeta Math

Proof. Here is the local model for p . Around the distinct points of a configuration choose pairwise disjoint small open disks $U_1, \dots, U_n \subset D^\circ$. Let V consist of configurations with exactly one point in each U_j . Then

$$p^{-1}(V) = \coprod_{\tau \in S_n} \tau \cdot (U_1 \times \dots \times U_n).$$

These $n!$ products are disjoint open sets. On each, forgetting labels is bijective onto V . It is a homeomorphism: the quotient map is open because for any open O , $p^{-1}(p(O)) = \bigcup_{\tau} \tau \cdot O$ is open. This proves that p is an $n!$ -sheeted covering, not just a many-to-one map. Compare the disjoint-neighbourhood proof. □

Consequently, an unordered path and an initial ordering determine unique continuous labelled trajectories. Locally, the point in each small disk inherits its label; when moving to the next neighbourhood, the existing labels determine which ordering to use.

Path and homotopy lifting

Theorem 3.C (Lifting and injectivity). *A path and its initial ordering determine a unique lift. Homotopies lift relative to a chosen initial lift. In particular, p_* is injective on fundamental groups.*

Alphabeta Math: path lifting · homotopy lifting · injectivity

Proof. For a path, pull back evenly covered neighbourhoods to an open cover of I . Compactness and the Lebesgue-number lemma give a finite subdivision so each subinterval maps into one neighbourhood. Choose its sheet using the initial value and continue across successive joins. Local inverse maps give continuity. Two lifts agreeing at one time agree near that time; the subdivision therefore proves uniqueness.

For a homotopy on I^2 , the same compactness argument gives a finite rectangular grid whose cells each map into an evenly covered neighbourhood. Work through the bottom row from left to right, then through successive rows. On a new cell select the sheet containing the already lifted bottom-left corner, and compose the homotopy on the whole cell with that sheet's inverse map. Its bottom edge, and its left edge when already constructed, agree with the existing lift by path uniqueness: both lift the same edge path and agree at the corner. Thus the construction preserves the prescribed bottom lift and all shared edges. The finite closed pasting lemma gives a continuous lift of the square. Uniqueness follows by applying path uniqueness on each vertical interval. This is the square case of the covering homotopy-lifting theorem; see also Hatcher, *Algebraic Topology*, §1.3.

In particular, the terminal ordering of a lifted based loop is unchanged under an endpoint-fixed homotopy: the terminal edge lifts to a continuous path in the finite discrete fibre, hence is constant. If a loop upstairs projects to a null-homotopic loop, lifting its contraction keeps both side edges at its initial point and the final edge constant. It therefore contracts upstairs too. Thus p_* is injective. This is the covering injection theorem, with its proof in the present setting. □

3.3 Braids as point motions

Definition 3.9 (Full and pure configuration braids). Define

$$B_n^{\text{conf}} = \pi_1(C_n(D^\circ), Q), \quad P_n = \pi_1(F_n(D^\circ), q).$$

In the full group, a loop returns to the same set Q ; its points may exchange places. In the **pure braid group**, every labelled point returns to its own q_j . The covering injection identifies P_n with a subgroup of B_n^{conf} . We retain the superscript until Chapter 5 proves the identification with the presented group from Chapter 2.

Alphabeta Math: full braids $\cdot$ pure braids

Definition 3.10 (Geometric braid). A geometric braid based at Q is the union of the graphs $\{(z_j(t), t) : 0 \leq t \leq 1\}$, where the z_j are continuous, pairwise distinct at every time, and their initial and terminal sets are Q . A braid isotopy is a continuous family of such motions fixing every strand endpoint.

Alphabeta Math: geometric braids $\cdot$ braid isotopy

Proposition 3.11 (Slicing and tracing). *Tracing a configuration loop and slicing a geometric braid give inverse identifications of their deformation classes, compatible with chronological concatenation.*

Alphabeta Math

Proof. Lift a based unordered loop γ to $(z_1(t), \dots, z_n(t))$ starting at q . Its trajectories

$$t \longmapsto (z_j(t), t) \quad \text{in } D^\circ \times I$$

are the strands of a **geometric braid**. Each meets every time slice exactly once. Two strands cannot intersect: an intersection would have the same time coordinate and then $z_i(t) = z_j(t)$, a forbidden collision. Conversely, slicing a braid given by these time-monotone trajectories gives a configuration loop. An endpoint-fixed configuration homotopy lifts to a continuous family of collision-free strands with their endpoints fixed; slicing such a family reverses the construction. Concatenation corresponds to stacking in chronological order. These inverse constructions identify configuration-loop classes with geometric braids modulo deformation through braids. Compare Alphabeta Math's comparison theorem: its geometric stacking uses the opposite order, so its raw slicing map reverses products. Here we explicitly use chronological stacking. The deformation is through time-monotone braids, rather than arbitrary link isotopy. □

Definition 3.12 (Positive elementary exchange). Fix $1 \leq i < n$, set $a = (q_i + q_{i+1})/2$, and $r = (q_{i+1} - q_i)/2$. Keep the other points fixed and move

$$z_i(t) = a - re^{\pi it}, \quad z_{i+1}(t) = a + re^{\pi it}.$$

Alphabeta Math

Proposition 3.13 (Elementary exchanges are configuration loops). *The preceding formulas define a counterclockwise loop σ_i in $C_n(D^\circ)$, whose ordered lift interchanges q_i and q_{i+1} .*

Alphabeta Math

Proof. Here $i^2 = -1$; its upright font distinguishes it from the index i . The disk of radius r about a lies in D° and contains no other q_j , while the moving points remain distance $2r$ apart. Both rotate counterclockwise: the initially left point passes below the real axis, and the initially right point passes above. At $t = 1$ they have exchanged places. Their unordered motion is a loop representing the positive generator σ_i ; the ordered motion is not a loop. □

Geometric interpretation. The cylinder is $D \times I$. Each horizontal slice contains the unordered configuration at that time; the full configuration space has dimension $2n$. For $n = 3$, the first two points turn counterclockwise around their midpoint and the third stays fixed. One exchange returns the set with two labels swapped; two exchanges return every label.

3.4 The permutation exact sequence

Theorem 3.14 (Full and pure braids). *For the inverse endpoint permutation $\rho([\gamma]) = e_\gamma^{-1}$, the sequence*

$$1 \longrightarrow P_n \xrightarrow{p_*} B_n^{\text{conf}} \xrightarrow{\rho} S_n \longrightarrow 1.$$

is exact. In particular, $B_n^{\text{conf}}/P_n \cong S_n$.

Alphabeta Math

Proof. For a lifted loop let $e_\gamma(j) = k$ mean that the strand starting at q_j ends at q_k . This forward endpoint permutation satisfies $e_{\alpha*\beta} = e_\beta \circ e_\alpha$, because we traverse α first. With the usual permutation multiplication $\tau v = \tau \circ v$, define instead

$$\rho([\gamma]) = e_\gamma^{-1}, \quad \rho([\alpha][\beta]) = \rho([\alpha])\rho([\beta]).$$

Thus ρ records, for each final position, the label that arrived there. Homotopy lifting proves that it depends only on $[\gamma]$. Each elementary exchange maps to the adjacent transposition $(i \ i+1)$. Adjacent transpositions generate S_n : successively swapping adjacent out-of-order entries sorts any list into $1, \dots, n$. Hence ρ is surjective.

A loop lies in $\ker \rho$ exactly when its lift ends at q , making it an ordered loop. Combined with the injectivity of p_* , this proves the **short exact sequence**

“Exact” here says that the first map is injective, its image is precisely the kernel of the second, and the second is onto. In particular P_n is normal and $B_n^{\text{conf}}/P_n \cong S_n$: two braids give the same permutation exactly when their quotient is pure. Explicitly, the map $gP_n \mapsto \rho(g)$ is well defined and injective because $\ker \rho = P_n$, is a homomorphism by the quotient multiplication, and is surjective because ρ is. This is the configuration braid exact sequence, with our inverse-endpoint convention made explicit.

□

Example: a nontrivial pure braid

Proposition 3.15 (Two elementary exchanges). *The element $[\sigma_i]^2$ is pure and nontrivial.*

Alphabeta Math: winding obstruction · covering injection

Proof. The square σ_i^2 lifts to a closed ordered loop: its two moving points rotate a full turn. It is nontrivial. Indeed the continuous map $F_n(D^\circ) \rightarrow S^1$ given by $z \mapsto (z_{i+1} - z_i)/|z_{i+1} - z_i|$ sends this loop to $t \mapsto e^{2\pi it}$. That circle loop cannot contract: under the covering $\mathbb{R} \rightarrow S^1$, $u \mapsto e^{2\pi iu}$, its lift starting at 0 ends at 1, whereas a constant loop ends at 0; lifted endpoints are homotopy-invariant. This map is a covering because any sufficiently short open arc has a continuous angle coordinate, and its inverse image consists of integer translates of that coordinate interval, each mapped homeomorphically to the arc. A contraction in ordered configuration space would therefore give an impossible contraction on the circle. Covering injectivity proves nontriviality in the full group too. The endpoint permutation detects purity, not all braiding.

□

3.5 Disk and plane models

Theorem 3.16 (Transport to the plane). *The map $h(z) = z/(1 - |z|)$ is a homeomorphism $D^\circ \rightarrow \mathbb{C}$, and it induces*

$$\pi_1(C_n(D^\circ), Q) \cong \pi_1(C_n(\mathbb{C}), h(Q)).$$

Alphabeta Math: configuration spaces · induced maps

Proof. The open disk and the plane are different spaces with an explicit homeomorphism:

$$h : D^\circ \longrightarrow \mathbb{C}, \quad h(z) = \frac{z}{1 - |z|}, \quad h^{-1}(w) = \frac{w}{1 + |w|}.$$

Both maps are continuous and inverse: the radius r becomes $r/(1 - r)$, whose inverse is $R/(1 + R)$, while the angle stays fixed. Applying h to every coordinate preserves distinctness and commutes with label permutations. It therefore gives homeomorphisms of ordered spaces and, by the quotient topology, of unordered spaces. Consequently

This is a transported-basepoint isomorphism, not equality of spaces. Other chosen planar base configurations give isomorphic groups after choosing connecting paths. Such paths exist: move labelled points one at a time through a finite set of unused staging positions, avoiding the finitely many stationary points. The plane minus finitely many points is path-connected, since a polygonal segment can detour around each forbidden point. Choose staging positions disjoint from both configurations, then move from staging to the target; applying h^{-1} gives the same fact in the open disk. □

The closed disk and basepoint transport

Proposition 3.D (Interior and closed-disk configurations). *Interior inclusion induces fundamental-group isomorphisms for ordered and unordered configurations, with basepoint transport as specified below.*

Alphabeta Math

Proof. The closed disk itself is not homeomorphic to the plane: it is compact and the plane is not. Nevertheless its configuration spaces have the same homotopy type as the interior versions. A **homotopy equivalence** is a continuous map with a continuous inverse up to homotopy: both composites are homotopic to the appropriate identity. For inclusion $j : D^\circ \hookrightarrow D$ and $s : D \rightarrow D^\circ$, $s(z) = z/2$, the induced configuration maps have this property. Coordinatewise radial scalings $T_u(z) = (1 - u/2)z$, $0 \leq u \leq 1$, give homotopies from the identities to the composites. All scaling factors are positive, so no distinct points collide.

The homotopies commute with permutations and descend continuously to unordered spaces. To check joint continuity, the map $p \times \text{id}_I$ is an open surjection: it takes each basic open product $O \times J$ to $p(O) \times J$, which is open since p is open. An open surjection is a quotient map. The continuous equivariant homotopy upstairs therefore defines a continuous homotopy downstairs by the definition of quotient topology.

Here the fundamental-group isomorphism can be proved directly at the same base configuration. In either the ordered or unordered model, write $b = q$ or $b = Q$, respectively, and let $c(u) = T_u(b)$, a path from b to $b/2$ lying in the interior configuration space. For any loop γ at b in the closed model, its scaling homotopy has side edges c . The two boundary routes of its parameter square give $\gamma * c \simeq c * (s \circ \gamma)$ with endpoints fixed: interpolate linearly between the bottom-then-right and left-then-top routes in the square and compose with the homotopy. Cancelling $c * \bar{c}$ gives $\gamma \simeq c * (s \circ \gamma) * \bar{c}$, an interior loop. Thus inclusion is surjective

on π_1 . If an interior loop contracts in the closed model, applying s gives an interior contraction of its scaled loop. The same square-route argument and cancellation recover an interior contraction of the original loop. Thus inclusion is injective as well. See the closed/interior disk comparison. We use interior points throughout to match the punctured-disk model of the next chapter.

□

3.6 Sources and further reading

Allen Hatcher, *Algebraic Topology*, Chapter 1, §1.3, Propositions 1.30–1.31 (printed pp. 60–61), gives the covering homotopy-lifting and fundamental-group injection theorems. The square proofs needed here are supplied above.

The directly linked Alfabeta Math items are also collected, with proofs, in The Fundamental Group, Covering Spaces and Lifting, Ordered and Unordered Configuration Spaces, and Braids as Fundamental Groups of Configuration Spaces. Their sign and multiplication conventions should be checked when comparing formulas; this series always traverses the first loop first and uses counterclockwise positive exchanges.

4 Mapping classes

Homeomorphisms + isotopies $\rightarrow$ Mapping class group $\rightarrow$ Chapter 6: compare with configuration loops

A braid moves points around a disk. A mapping class records how the disk itself has been rearranged, allowing deformations that respect its boundary and marked points. This chapter defines that group. The proof that it is the braid group comes in Chapter 6.

Prerequisites: homeomorphisms, orientation, and quotients

A **homeomorphism** is a continuous bijection with continuous inverse. An orientation-preserving homeomorphism respects the usual counterclockwise orientation of the plane. No differentiability is required. Composition $f \circ g$ applies g first.

A quotient group G/N identifies elements differing by an element of a normal subgroup N . Normality means $gNg^{-1} = N$. See Alphabet Math's normal subgroups and quotient groups. We prove the needed normality below.

Why an isotopy preserves orientation

Orientation can be tested by winding number. For an interior point x , choose a small positively oriented circle c about x contained in the disk. A homeomorphism f preserves orientation when, for every such interior point and small circle, the loop $(f(c(t)) - f(x))/|f(c(t)) - f(x)|$ has winding number $+1$. Winding number is the endpoint difference of a lift to $\mathbb{R}$ under $v \mapsto e^{2\pi i v}$; see the covering argument in Chapter 3. For a continuous family of homeomorphisms, the normalized difference stays defined because no circle point maps to the image of its centre. It gives a homotopy of circle loops, so its integer winding number stays constant by homotopy lifting. Consequently a homeomorphism joined to the identity is orientation preserving. This is the only orientation-invariance fact used for our explicit constructions.

4.1 Boundary-fixed homeomorphisms

Fix $n \geq 3$. In the complex plane, put

$$D = \{z \in \mathbb{C} : |z| \leq 1\}, \quad q_j = -1 + \frac{2j}{n+1} \quad (1 \leq j \leq n), \quad Q = \{q_1, \dots, q_n\}.$$

The points of Q lie in the interior. The **punctured disk** is $D \setminus Q$. We define its mapping classes using the marked disk: keep the points in the disk and specify how maps treat them. Every allowed map restricts to a homeomorphism of $D \setminus Q$, because it preserves Q setwise; the allowed punctured-disk isotopies are these ambient isotopies restricted to $D \setminus Q$.

Definition 4.1 (The marked-disk homeomorphism group). Let

$$\mathcal{H}_Q = \{f : D \rightarrow D \mid f \text{ is an orientation-preserving homeomorphism,} \\ f|_{\partial D} = \text{id}, \quad f(Q) = Q\}.$$

The boundary condition is **pointwise**: every boundary point stays exactly where it was. The puncture condition is **setwise**: f may permute the q_j . These are different requirements. A rigid rotation of the whole disk generally violates the first, even though it carries the boundary circle onto itself.

Alphabet Math

Proposition 4.2 (Composition and inversion). *The set $\mathcal{H}_Q$ is a group under ordinary composition.*

Alphabeta Math

Proof. Composition makes $\mathcal{H}_Q$ a group. Composites and inverses are orientation preserving, fix every boundary point, and carry Q onto itself. The identity map satisfies all three conditions. □

4.2 Isotopies and mapping classes

Definition 4.3 (Boundary-fixed marked isotopy). An **isotopy in $\mathcal{H}_Q$** from f to g is a continuous map $H : D \times [0, 1] \rightarrow D$ such that $H_t(z) = H(z, t)$ belongs to $\mathcal{H}_Q$ for every t , with $H_0 = f$ and $H_1 = g$. Thus each intermediate map is a homeomorphism, fixes the boundary pointwise, and preserves the marked set.

A **homotopy** requires continuous intermediate maps, which need not be homeomorphisms. It can collapse or fold the disk. Our equivalence relation uses isotopies with the stated boundary and puncture conditions. Reversing and concatenating isotopies, along with constant isotopies, proves that this is an equivalence relation.

Alphabeta Math

Lemma 4.4 (Stationary marked-point images). *An isotopy in $\mathcal{H}_Q$ has a time-independent permutation of Q . An identity isotopy fixes each marked point throughout.*

Alphabeta Math

Proof. For each j , the map $t \mapsto H_t(q_j)$ is continuous and takes values in the finite discrete set Q . Since $[0, 1]$ is connected, that map is constant:

$$H_t(q_j) = f(q_j) \quad \text{for every } t.$$

Consequently isotopic maps induce the same puncture permutation. If $H_0 = \text{id}$, each puncture stays at its own location throughout the isotopy. Requiring setwise preservation at every time already forces this; it does not let punctures travel between marked locations. □

Definition 4.5 (Mapping class group). Write $[f]$ for the isotopy class of f . The **mapping class group** is

$$\text{Mod}(D, Q; \partial D) = \mathcal{H}_Q / \sim.$$

This is the boundary-fixed convention in Alphabeta Math's definition and in González-Meneses, §1.4. The boundary notation is part of the definition.

Alphabeta Math

4.3 Group operations on mapping classes

Theorem 4.6 (The mapping class group law). *The operations $[f][g] = [f \circ g]$ and $[f]^{-1} = [f^{-1}]$ are well defined on isotopy classes and make $\text{Mod}(D, Q; \partial D)$ a group.*

Alphabeta Math

Proof. Suppose f_t joins f to f' , and g_t joins g to g' , within $\mathcal{H}_Q$. Then $f_t \circ g_t$ joins $f \circ g$ to $f' \circ g'$. The inverse family f_t^{-1} joins f^{-1} to $(f')^{-1}$. Therefore

$$[f][g] = [f \circ g], \quad [f]^{-1} = [f^{-1}]$$

are independent of representatives. Associativity and the identity law follow from those for maps. In this multiplication the right factor acts first. $\square$

Prerequisite: topology on the homeomorphism group

Proposition 4.A (Uniform topology and inverse families). *On the compact disk, the compact-open and uniform topologies agree. Composition and inversion are continuous. Paths of homeomorphisms are equivalent to jointly continuous isotopies.*

Alphabeta Math: uniform topology · homeomorphism group

Proof. The map $\Phi : D \times [0, 1] \rightarrow D \times [0, 1]$, $\Phi(x, t) = (f_t(x), t)$, is a continuous bijection. Its domain is compact and its target is Hausdorff, so it is a homeomorphism. Its inverse is $(y, t) \mapsto (f_t^{-1}(y), t)$. This proves the required joint continuity without assuming differentiability.

For the topology on the group, use the uniform distance $d(f, g) = \sup_{x \in D} |f(x) - g(x)|$. On the compact disk this gives the compact-open topology. The compact-open topology has subbasic sets $\{g : g(K) \subset V\}$, with $K \subset D$ compact and $V \subset D$ open. Such a set is uniformly open: the compact set $f(K) \subset V$ has positive distance from the closed complement of V , unless that complement is empty. Conversely, given $\varepsilon > 0$, choose finitely many compact neighbourhoods K_j covering D with $|f(x) - f(x_j)| < \varepsilon/3$ on K_j . The finitely many conditions $g(K_j) \subset B(f(x_j), 2\varepsilon/3) \cap D$ give a compact-open neighbourhood of f contained in its uniform ε -ball. A jointly continuous isotopy is uniformly continuous on $D \times [0, 1]$, so $t \mapsto f_t$ is a path continuous in the uniform topology. Conversely, continuity of this path and continuity of each f_t give joint continuity by

$$|f_t(x) - f_{t_0}(x_0)| \leq d(f_t, f_{t_0}) + |f_{t_0}(x) - f_{t_0}(x_0)|.$$

For a fixed continuous map $u : D \rightarrow D$, define its modulus of continuity by

$$\omega_u(\delta) = \sup\{|u(x) - u(y)| : x, y \in D, |x - y| \leq \delta\}.$$

Compactness implies $\omega_u(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Inserting $f \circ g'$ between two composites gives

$$d(f' \circ g', f \circ g) \leq d(f', f) + \omega_f(d(g', g)).$$

For inversion, put $x = g^{-1}(y)$. Then

$$|f(x) - y| = |f(x) - g(x)| \leq d(f, g)$$

Therefore

$$|g^{-1}(y) - f^{-1}(y)| = |f^{-1}(f(x)) - f^{-1}(y)| \leq \omega_{f^{-1}}(d(f, g)).$$

Taking the supremum over y proves $d(g^{-1}, f^{-1}) \leq \omega_{f^{-1}}(d(f, g))$. Thus composition and inversion are continuous: both the unmarked boundary-fixed homeomorphism group and its marked-set stabilizer $\mathcal{H}_Q$ are topological groups. This is the topology used in Chapter 6. $\square$

Proposition 4.7 (Quotient by the identity path component). *The identity path component $\mathcal{H}_{Q,0}$ is normal in $\mathcal{H}_Q$, and $\text{Mod}(D, Q; \partial D) = \mathcal{H}_Q / \mathcal{H}_{Q,0}$.*

Alphabeta Math

Proof. There is also a quotient-by-subgroup description. Let $\mathcal{H}_{Q,0}$ be the maps isotopic to the identity in $\mathcal{H}_Q$. With the compact-open topology, an isotopy is exactly a path of homeomorphisms. Thus $\mathcal{H}_{Q,0}$ is the **identity isotopy component**; “component” here means path component.

It is a subgroup because products and inverses of identity isotopies are identity isotopies. It is normal: if h_t joins the identity to h , then $f \circ h_t \circ f^{-1}$ joins the identity to $f \circ h \circ f^{-1}$, still inside $\mathcal{H}_Q$. Finally, $f \sim g$ exactly when $g^{-1} \circ f \in \mathcal{H}_{Q,0}$, by composing the relevant isotopy with g or g^{-1} . Hence

$$\text{Mod}(D, Q; \partial D) = \mathcal{H}_Q / \mathcal{H}_{Q,0}.$$

□

4.4 Pure mapping classes

Definition 4.8 (The puncture permutation and pure subgroup). Define $\rho([f]) \in S_n$ by $f(q_j) = q_{\rho([f])(j)}$, where S_n is the group of permutations of $\{1, \dots, n\}$. The stationary-puncture argument makes ρ well defined. Under ordinary permutation composition,

$$\rho([f][g]) = \rho([f]) \circ \rho([g]).$$

The **pure mapping class group** is the kernel $\text{PMod}(D, Q; \partial D) = \ker \rho$. Its representatives fix each q_j individually. The full group permits permutations. Fixing each puncture at the endpoint does not imply that a map is isotopic to the identity; it only makes its class pure.

Alphabeta Math

4.5 Supported half twists

A local rotation interchanges a pair of punctures. A radial cutoff makes this rotation compatible with pointwise fixation of the outer boundary.

Definition 4.9 (A clockwise half-twist motion). Choose consecutive punctures q_i, q_{i+1} . Set $c = (q_i + q_{i+1})/2$ and $a = 1/(n+1)$. Choose radii $a < r_0 < R < 2a$. The disk $|z - c| \leq R$ lies inside D and contains exactly this pair: the next puncture is $3a$ from c , and the closest outer boundary is at least $3a$ away.

Let $b(r)$ be a continuous radial cutoff equal to 1 for $r \leq r_0$, decreasing linearly to 0 on $[r_0, R]$, and equal to 0 for $r \geq R$. Define

$$h_t(c + re^{i\theta}) = c + re^{i(\theta - \pi tb(r))}, \quad 0 \leq t \leq 1.$$

Alphabeta Math

Proposition 4.10 (The half-twist endpoint). *Every h_t in the preceding definition is a boundary-fixed orientation-preserving homeomorphism. The endpoint h_1 exchanges the chosen pair and fixes all other punctures. For $0 < t < 1$, the map h_t does not preserve Q .*

Alphabeta Math

Proof. At $r = 0$, set $h_t(c) = c$. Radius is unchanged, and the inverse adds the same angle instead of subtracting it. Each h_t is therefore a homeomorphism, joined to the identity through homeomorphisms and hence orientation preserving. It is the identity outside the support disk, so it fixes the outer boundary. At $t = 1$ it exchanges the two punctures clockwise and fixes every other puncture.

For every $0 < t < 1$, the moving pair is outside Q : both points have nonzero imaginary coordinate, whereas all marked points are real. Thus h_t is an isotopy of the *unmarked* disk, but not a permitted isotopy from the identity to h_1 in $\mathcal{H}_Q$. Indeed, h_1 cannot be isotopic to the identity there because its puncture permutation is nontrivial. See also the supported half-twist example. □

Geometric interpretation. The supported clockwise half-twist motion is h_t . Its point tracks are $(h_t(q_j), t)$, and it fixes the boundary pointwise. The counterclockwise motion h_t^{-1} lifts the positive configuration loop, whose inverse endpoint represents $[h_1]$. Intermediate maps generally move the marked set; the endpoints define the mapping classes.

Theorem 4.11 (The puncture-permutation exact sequence). *The sequence*

$$1 \longrightarrow \text{PMod}(D, Q; \partial D) \longrightarrow \text{Mod}(D, Q; \partial D) \xrightarrow{\rho} S_n \longrightarrow 1.$$

is exact.

Alphabeta Math

Proof. The classes $[h_1]$ realize every adjacent transposition $(i \ i+1)$. Adjacent transpositions generate S_n : swapping adjacent out-of-order entries repeatedly sorts any permutation into increasing order. Thus ρ is surjective. We have proved the exact sequence

Its kernel is the pure subgroup by Definition 4.8. □

4.6 Dehn twists and the boundary condition

Definition 4.12 (Dehn twist on an annulus). In annular coordinates, a clockwise Dehn twist is the homeomorphism $(r, \theta) \mapsto (r, \theta - 2\pi b(r))$, where the continuous cutoff b equals 1 at one boundary and 0 at the other. Both boundary circles are fixed; extend the map by the identity beyond the annulus.

Alphabeta Math: related annular twist

Proposition 4.13 (The square of a half twist). *The map h_1^2 is a clockwise Dehn twist about a curve enclosing the chosen puncture pair. Under the equivalences of Chapters 5 and 6, its class corresponds to s_i^2 and has infinite order.*

Alphabeta Math: supported half twist · exponent sum

Proof. The square h_1^2 rotates the inner disk by -2π , so its endpoint map is the identity there. On the surrounding annulus it makes one full angular turn across the annulus, and outside it is the identity. This is a **Dehn twist** about a curve enclosing the pair, with clockwise sign in these coordinates. Its permutation is trivial. Here a Dehn twist means an annulus map $(r, \theta) \mapsto (r, \theta - 2\pi b(r))$, extended by the identity beyond its boundary circles; the two boundary values of b are 1 and 0, so both circles are fixed. This directly identifies the representative. After the isomorphisms in Chapters 5 and 6, $[h_1^2]$ corresponds to the abstract element s_i^2 . The exponent-sum homomorphism of Chapter 2 sends s_i^{2m} to $2m$, so this class has infinite order. Its trivial permutation alone does not establish that fact. □

Proposition 4.14 (A boundary twist and a rotating boundary). *A full clockwise twist about all punctures fixes the outer boundary pointwise. It becomes isotopic to the identity when the boundary is only required to be preserved setwise.*

Alphabeta Math

Proof. Likewise, choose $\max_j |q_j| < r_0 < R < 1$, and now use a radial cutoff centred at 0, equal to 1 up to r_0 and 0 from R onward. The endpoint $T(re^{i\theta}) = re^{i(\theta - 2\pi b(r))}$ is a full twist supported in the annulus $r_0 \leq r \leq R$ near the boundary. It fixes the actual boundary pointwise. A rigid rotation is not such a representative. If the boundary were only preserved setwise, a compensating boundary rotation would undo it. For a cutoff b equal to 1 on the inner disk and 0 near the boundary, the maps $(r, \theta) \mapsto (r, \theta - 2\pi t(b(r) - 1))$ fix the punctures throughout, rotate the boundary for intermediate t , and at $t = 1$ give exactly the same full-twist endpoint. This is an allowed isotopy only under the setwise boundary convention. See the boundary-setwise counterexample and González-Meneses, §1.4, p. 7. Under Chapter 6's inverse-endpoint correspondence, the clockwise boundary-twist class is the image of the loop rotating all marked points counterclockwise through 2π . Indeed the lifted motion $(r, \theta) \mapsto (r, \theta + 2\pi tb(r))$ moves those points in exactly this way, and its inverse endpoint is our twist. □

Corollary: nontriviality of the boundary twist

Corollary 4.15 (Infinite order of the boundary twist). *Under the pointwise boundary convention, the full boundary-twist class has infinite order.*

Alphabeta Math: full-twist loop · winding obstruction

Proof. After Chapter 6 proves that the inverse-endpoint map is injective, the boundary twist has infinite order. Its preimage is the ordered loop $z_j(t) = e^{2\pi it} q_j$, projected to unordered configurations. For any distinct pair j, k , the map $(z_1, \dots, z_n) \mapsto (z_k - z_j)/|z_k - z_j|$ sends its m -th power to a circle loop of winding number m . Such a loop contracts only if $m = 0$, by the circle-covering argument of Chapter 3. The covering injection from ordered to unordered configuration groups and the Chapter 6 isomorphism transfer this conclusion to $[T]^m$. Thus the explicit isotopy with rotating boundary really kills a nontrivial class when the pointwise boundary condition is relaxed. □

Remark 4.16 (Compatibility of sign and multiplication). Our configuration-space generator in Chapter 3 exchanges the pair **counterclockwise**: the initially left point travels below, and the initially right point above. For configuration loops, $\alpha * \beta$ means α first, then β . Mapping classes multiply by usual composition, with the right factor first. To reconcile those orders, Chapter 6 sends a lifted motion F_t to $[F_1^{-1}]$. Thus that positive braid generator σ_i corresponds to the **clockwise** half-twist class constructed here. Endpoint assignment $[F_1]$ alone reverses product order; it must not silently be called the same homomorphism.

Alphabeta Math

4.7 Sources and further reading

- Juan González-Meneses, *Basic results on braid groups*, §1.4, printed pp. 6–7: the boundary-fixed mapping-class model and the role of the boundary twist. This chapter specifies its own multiplication and sign conventions.
- Alphabeta Math: Punctured disks, mapping classes, and point pushing, with the directly linked definition and examples above.

5 The Artin isomorphism

Artin presentation $\rightarrow$ Explicit generator loops $\rightarrow$ Crossing words and local relations $\rightarrow$ All homotopy events $\rightarrow$ Isomorphism

The Artin relations describe both elementary exchanges and every change of crossing word under homotopy. The second assertion is the completeness part of the presentation.

The other input is Chapter 3's configuration-space model. The technical general-position prerequisite is stated in the expandable section below; the braid argument after that prerequisite is proved here.

5.1 The comparison homomorphism

Fix $n \geq 3$. Let $D = \{z \in \mathbb{C} : |z| \leq 1\}$, let D° be its interior, and put $q_j = -1 + 2j/(n+1)$ and $Q = \{q_1, \dots, q_n\}$. Write

$$F_n(D^\circ) = \{(z_1, \dots, z_n) \in (D^\circ)^n : z_j \neq z_k \text{ for } j \neq k\}, \quad C_n(D^\circ) = F_n(D^\circ)/S_n.$$

Let A_n denote the abstract group

$$\langle s_1, \dots, s_{n-1} \mid s_i s_j = s_j s_i \ (|i - j| \geq 2), \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \ (1 \leq i \leq n-2) \rangle.$$

Theorem 5.1 (The Artin isomorphism). *There is an isomorphism $\Phi : A_n \rightarrow \pi_1(C_n(D^\circ), Q)$ determined by $\Phi(s_i) = [\sigma_i]$. Here σ_i is the counterclockwise elementary exchange.*

Alphabeta Math: Artin completeness · configuration comparison

Products of loops traverse the first loop *then* the second. For $c_i = (q_i + q_{i+1})/2$ and $r = 1/(n+1)$, define a loop σ_i of unordered configurations by fixing all other points and moving these two:

$$z_i(t) = c_i - re^{\pi i t}, \quad z_{i+1}(t) = c_i + re^{\pi i t}, \quad 0 \leq t \leq 1.$$

Here $i^2 = -1$, while the subscript i indexes the pair. Each point rotates counterclockwise about c_i . The point starting on the left passes below, and the point starting on the right passes above. The two points stay distance $2r$ apart; their circle lies in D° , and no fixed point lies on it. At $t = 1$ their positions have exchanged, so the unordered configuration returns to Q . The intended map is $\Phi(s_i) = [\sigma_i]$. We must first check that it respects the defining relations.

The proof of Theorem 5.1 follows after the crossing-word and homotopy lemmas below.

5.2 Crossing words

Definition 5.2 (A generic crossing word). Lift a loop to labeled paths $z_j(t) = x_j(t) + iy_j(t)$, starting at $(q_1, \dots, q_n)$. Draw $(x_j(t), t)$, with time increasing upward. At a projected crossing, declare the strand with larger y to pass over the other. A crossing is positive when the strand immediately to its left before the crossing has smaller y ; this makes the loop above positive. Record s_i or s_i^{-1} , where $i, i+1$ are the crossing strands' positions in the current left-to-right order, not their original labels.

A generic loop has finitely many crossing times, each with exactly one equality $x_j = x_k$, with $\frac{d}{dt}(x_j - x_k) \neq 0$. The crossing strands are adjacent in real-coordinate order. Because $z_j \neq z_k$, their imaginary coordinates differ there, so the sign is defined.

Alphabeta Math

Lemma 5.3 (Reconstruction from crossings). *A generic loop is homotopic to the chronological product of elementary exchanges recorded by its crossing word.*

Alphabet Math

Proof. The open set $U \subset C_n(D^\circ)$ of configurations with distinct real coordinates has a unique left-to-right ordering. In those coordinates it is convex: the disk conditions and $x_1 < \dots < x_n$ are convex. Thus every path segment in U can be straightened, and every connector to Q is unique up to homotopy.

Near one crossing, choose labels for the two crossing points and require all other real-coordinate inequalities to remain strict, allowing only this adjacent pair to change order. Also impose the fixed strict inequality between their imaginary coordinates. The resulting set W is convex in the labeled configuration space. It contains no collision: unequal real coordinates separate every other pair, and the imaginary inequality separates this pair. Both side intersections with the corresponding chambers are convex. Hence any passage through W , closed using chamber connectors, represents the same loop as the appropriate $\sigma_i^{\pm 1}$. Use a truncated passage around the semicircle's crossing, so its endpoints also satisfy the strict imaginary inequality; the remaining arcs lie in the real-order chambers. One can move the crossing configuration to this passage while preserving the inequalities; convexity gives the required homotopies. Subdivide the loop into chamber segments and single-crossing passages. Its class is the product of the recorded generators. $\square$

5.3 The Artin relations

Lemma 5.4 (Commutation and the three-strand relation). *The elementary-loop classes satisfy the distant commutation and adjacent Artin relations.*

Alphabet Math: three-strand relation · Artin relations

Proof. For $|i - j| \geq 2$, carry out the two exchanges in disjoint small disks about their respective midpoint circles. Allow their progress parameters to vary independently over a square. Every point in that square is a configuration without collisions. The two boundary routes perform the exchanges in opposite orders, proving $[\sigma_i][\sigma_j] = [\sigma_j][\sigma_i]$.

For the adjacent relation, isolate three consecutive points in an interior disk containing no other point. Fix their mean real coordinate at its centre and keep their imaginary coordinates at three small distinct heights $y_A < y_B < y_C$. In their real coordinates, the walls $x_A = x_B$, $x_A = x_C$, and $x_B = x_C$ are three lines through the origin in the two-dimensional difference plane. Take a small disk around their common intersection in this difference plane. Choosing both the heights and the disk radius sufficiently small keeps all three points inside the interior disk, with the other points fixed and separated in real coordinate. Throughout this disk the three points remain distinct because their heights are distinct. The two semicircular routes from the chamber A, B, C to the opposite chamber C, B, A cross the walls in orders AB, AC, BC and BC, AC, AB . All six crossings are positive by the stated height order. Close the routes using paths in U . The filled disk and chamber contractions give

$$[\sigma_i][\sigma_{i+1}][\sigma_i] = [\sigma_{i+1}][\sigma_i][\sigma_{i+1}].$$

The disk may be chosen inside a small disk in D° , with all other points fixed. This proves the relation by an actual configuration-space homotopy. Consequently $\Phi : A_n \rightarrow \pi_1(C_n(D^\circ), Q)$ is well defined. The preceding crossing-word argument proves it is surjective. $\square$

Corollary 5.5 (A surjective comparison homomorphism). *The assignment $s_i \mapsto [\sigma_i]$ defines a surjective homomorphism $\Phi : A_n \rightarrow \pi_1(C_n(D^\circ), Q)$.*

Proof. Lemma 5.4 verifies the relators, so the presentation universal property in Chapter 2 defines Φ . Lemma 5.6 provides a generic representative of each loop; Lemma 5.3 expresses that representative as a product of generator images. □

Prerequisites and proof of general position

Lemma 5.6 (Generic representatives and homotopies). *Every configuration loop has a generic representative. An endpoint-fixed homotopy between generic representatives may be chosen so that its crossing word changes at finitely many distinct parameters, each by a pair cancellation, a disjoint pair interchange, or a triple crossing.*

Alphabeta Math: generic representatives · transversality

Proof. The differential-topology input consists of smooth approximation and transversality. We import these theorems from Hirsch, *Differential Topology*, Chapters 2 and 3, in their relative forms. The finite jet-transversality consequence needed here is explained below; no multijet theorem is needed. The following definitions and theorem statement specify the input; the application to braids is then checked below.

Smooth approximation means replacing a continuous map into an open subset of Euclidean space by an arbitrarily uniformly close map with derivatives of every order. If the original map is already smooth near a specified closed region, the replacement can agree there. For our parameter square, we use boundary collars as that region. This replacement is homotopic to the original whenever the straight interpolation remains in the target.

A smooth map $f : M \rightarrow N$ is **transverse** to a smooth submanifold $S \subset N$ if, whenever $f(p) \in S$,

$$Df_p(T_p M) + T_{f(p)} S = T_{f(p)} N.$$

Here $T_p M$ is the vector space of infinitesimal directions at p , and Df_p is the derivative. Equivalently, variations of the input supply every direction normal to S . If S is defined locally by c independent equations, transversality says their derivatives after composition with f have rank c . Its inverse image is then a smooth set of dimension $\dim M - c$; it is empty if $c > \dim M$. The number c is the **codimension** of S .

An **r -jet** records a map's value and all its derivatives through order r at one source point, together with that point. Thus a 2-jet of a real function on our square records $f, f_s, f_t, f_{ss}, f_{st}, f_{tt}$.

Imported transversality theorem. For a smooth family of maps whose joint evaluation is transverse to a smooth submanifold, almost every parameter gives a transverse map (parametric transversality, a consequence of Sard's theorem). Smooth maps transverse to finitely many specified smooth submanifolds can therefore be obtained by arbitrarily small perturbations, fixed where the conditions already hold. We use this relative theorem and smooth approximation from Hirsch, Chapters 2–3.

Here is why this also applies to our finite list of jet conditions. On a small source disk, perturb each coordinate by a polynomial of degree at most two times a smooth bump equal to one on a smaller disk. Its coefficients independently vary the value, first derivatives, and second derivatives at every point of that smaller disk. Thus the joint map from coefficients and source points into 2-jets is a submersion, hence transverse to every jet stratum. Finitely many smaller disks cover the compact region needing perturbation; combine their coefficient families and apply parametric transversality to the finitely many strata. Bumps are supported away from the fixed collars. On the remaining region the required conditions already

hold and persist under sufficiently small perturbations. This establishes the special relative jet consequence we use, rather than importing braid genericity as a theorem.

A homotopy $H(s, t)$, $0 \leq s, t \leq 1$, of based loops lifts to $F_n(D^\circ)$ because the square is simply connected and $F_n \rightarrow C_n$ is a covering. The lift has the fixed starting tuple on $t = 0$ and a fixed permuted tuple on $t = 1$; the permutation is locally constant in s , hence constant. First reparameterize time by a smooth nondecreasing map constant near 0 and 1 and strictly increasing between those collars. This produces constant endpoint collars by an endpoint-fixed homotopy. Approximate each boundary loop smoothly relative to these collars, then perturb its real coordinates transverse to the equality strata. These close approximations are homotopic to the original loops by straight interpolation upstairs. Attach those homotopies to the given square. Reparameterizing the square's homotopy coordinate similarly makes it independent of s near $s = 0, 1$, equal there to the chosen smooth generic loops. Reparameterize its time coordinate to retain the constant endpoint collars. The square is now smooth on all four boundary collars: the t -collars have distinct real coordinates, and on the s -collars every crossing has nonzero t -derivative. Thus the relative approximation and perturbation hypotheses really hold there.

Compactness supplies a positive margin

$$\delta = \min_{s,t} \min \left(\min_j (1 - |z_j(s, t)|), \min_{j < k} |z_j(s, t) - z_k(s, t)| \right) > 0.$$

A uniform perturbation of each coordinate by less than $\delta/4$, and its straight interpolation with the original lift, stays in the disk and avoids collisions. Thus approximation and perturbation preserve the homotopy class. All subsequent transversality perturbations can be chosen this small and fixed on the prescribed boundary collars.

Stratify the real-coordinate space by partitions into sets of equal coordinates. A block of size m imposes $m - 1$ independent equations. Transversality of a one-dimensional path avoids strata of codimension at least two and makes every codimension-one encounter transverse. Transversality of the two-dimensional homotopy avoids strata of codimension at least three. Its isolated codimension-two encounters therefore have either two disjoint equal pairs, or one triple of equal real coordinates. The triple's three pairwise equations have rank two, not three.

On a single-pair wall, write $f = x_j - x_k$. Transversality makes $f = 0$ a smooth curve in the parameter square. Apply jet transversality also to the two equations $f = f_t = 0$. At such a point the derivative matrix of (f, f_t) has determinant $f_s f_{tt}$, so its required rank two gives $f_s \neq 0$ and $f_{tt} \neq 0$. These are ordinary folds, locally $s - s_0 = \pm(t - t_0)^2$ after smooth coordinate changes preserving the slicing parameter. Indeed, the implicit-function theorem writes the wall as a graph $s = g(t)$, with $g'(t_0) = 0$ and $g''(t_0) \neq 0$. Taylor's formula gives $g(t) - g(t_0) = (t - t_0)^2 h(t)$ with smooth nonzero h ; absorbing $\sqrt{|h(t)|}$ into the time coordinate proves the fold form.

At a codimension-two intersection, both independent wall equations have independent differentials. Require also that each pairwise difference has nonzero t -derivative. At a triple, this includes all three pairs. Failures add a condition and have codimension at least three. Consequently the wall arcs meet transversely and their crossing times exchange order as s passes the event.

There are finitely many events. Each codimension-two intersection is isolated and belongs to a closed compact zero set. Fold points are likewise isolated; they cannot accumulate at intersections because every pairwise t -derivative there is nonzero. A limit at a single-pair wall is another fold, where the rank-two condition again gives isolation. The fixed collars contain no events. Compactness therefore makes the entire event set finite.

Separate their s -values without a further transversality theorem. Choose disjoint tiny

neighborhoods of these finitely many parameter points and bumps β_r equal to one near each point and zero outside its neighborhood. The change of source coordinates

$$(s, t) \mapsto \left(s + \sum_r \epsilon_r \beta_r(s, t), t \right)$$

is a diffeomorphism fixed on the boundary collars when the constants ϵ_r are sufficiently small. Compose the homotopy with its inverse. Near event r , this merely translates its s -coordinate by ϵ_r . On the transition regions every wall has noncritical projection to s ; compactness and a sufficiently small first-derivative change preserve that fact. Thus no new fold appears, intersections are simply carried along, and their s -values can be chosen distinct.

Each nonexceptional time slice has finitely many transverse crossings: their zero sets are compact and isolated, with none on the endpoint collars. Between event parameters the implicit-function theorem follows each crossing time continuously; crossings cannot appear, disappear, or exchange order there. Their imaginary differences cannot change sign on a wall without a forbidden actual collision. The crossing word is therefore constant between events. This proves the exhaustive list, using only the stated smooth-approximation and parametric-transversality input. □

5.4 Completeness of the relations

Lemma 5.7 (Homotopy invariance of the Artin word). *Homotopic generic loops have crossing words defining the same element of A_n .*

Alphabeta Math

Proof. Now let two generic based loops be homotopic. Apply the preceding prerequisite to an endpoint-fixed homotopy. Sweeping s from 0 to 1, there are exactly three possible kinds of change.

1. **A fold creates or removes two crossings of the same pair.** No other strand is between the pair locally, so their index is the same. Their imaginary order is unchanged, while their crossing directions are opposite. The subword is $s_i s_i^{-1}$ or $s_i^{-1} s_i$, which cancels.
2. **Two disjoint pairs cross at the same time.** Their real-coordinate locations are distinct; otherwise four points would coincide in real coordinate, a forbidden codimension-three event. The pairs occupy disjoint adjacent positions, so $|i - j| \geq 2$. Their crossing times exchange order, giving $s_i^\epsilon s_j^\eta \leftrightarrow s_j^\eta s_i^\epsilon$. These identities, including negative exponents, follow from $s_i s_j = s_j s_i$.
3. **Three strands have the same real coordinate.** Their imaginary coordinates are pairwise distinct, since actual collisions are excluded. The nonzero pairwise time derivatives mean that the three real-coordinate velocities are distinct. Thus their real ordering reverses across the event. The following calculation verifies both the crossing orders and their signs for every orientation of the event.

There is no Reidemeister-I event. Every strand is a graph over time, so a strand never intersects itself in the (x, t) projection and never develops the local time maximum or minimum required for a curl. “Projected coincidence” in this proof always means equal real coordinates of different points; it never means that the points collide in the disk.

The six possible triple signs

Choose a short time interval about the event and label the strands A, B, C in their real order at its beginning. At the event their distinct velocities satisfy $v_A > v_B > v_C$: just before it, $x_A - x_B$ and $x_B - x_C$ are negative. Each pair crosses exactly once on nearby slices by the implicit-function theorem, and the order at the interval's end is C, B, A . Every crossing is adjacent. The only possible three-step reversals are consequently AB, AC, BC and BC, AC, AB . To see that the two sides give different orders, write the crossing-time graphs as $t = \tau_{AB}(s)$ and $t = \tau_{BC}(s)$. Independence of the two wall differentials gives $\tau'_{AB}(s_0) \neq \tau'_{BC}(s_0)$, so their times exchange order at s_0 . The AC crossing lies between them: $x_A - x_C = (x_A - x_B) + (x_B - x_C)$, with both summands increasing in time near the event. This verifies the three-crossing move for either direction of the s -sweep and any original labeling.

Put $a = s_i$, $b = s_{i+1}$, and

$$p = \operatorname{sgn}(y_B - y_A), \quad q = \operatorname{sgn}(y_C - y_A), \quad r = \operatorname{sgn}(y_C - y_B).$$

The local move is

$$a^p b^q a^r \longleftrightarrow b^r a^q b^p.$$

Since the imaginary heights have a total order, only six of the eight sign triples can occur:

Every possible height order and its word identity		
Heights, lowest first	(p, q, r)	Identity
A, B, C	$(+, +, +)$	$aba = bab$
A, C, B	$(+, +, -)$	$aba^{-1} = b^{-1}ab$
B, A, C	$(-, +, +)$	$a^{-1}ba = bab^{-1}$
B, C, A	$(-, -, +)$	$a^{-1}b^{-1}a = ba^{-1}b^{-1}$
C, A, B	$(+, -, -)$	$ab^{-1}a^{-1} = b^{-1}a^{-1}b$
C, B, A	$(-, -, -)$	$a^{-1}b^{-1}a^{-1} = b^{-1}a^{-1}b^{-1}$

These are all consequences of one relation. From $aba = bab$, multiply on the left by b^{-1} and on the right by a^{-1} to get the second identity. Multiplying instead on the left by a^{-1} and on the right by b^{-1} gives the third. Their inverses give the fifth and fourth, respectively, and inverting the original relation gives the sixth. The excluded triples $(+, -, +)$ and $(-, +, -)$ would require a cyclic ordering of three real heights, which is impossible.

All these moves take place in genuine configurations: near a triple event the distinct imaginary heights remain separated; near a double event the two crossing pairs remain separate; near a fold the crossing pair has distinct heights. Choosing sufficiently small parameter neighborhoods preserves these facts and the separation from the other points. $\square$

5.5 Proof of the Artin isomorphism

Proof of Theorem 5.1. Denote the crossing-word map by Ω , reserving Ψ for chapter 6. Assign to a loop its generic crossing word, interpreted in A_n . A different generic approximation is homotopic to it, and the exhaustive event list shows that homotopic loops give equal elements of A_n . Thus this defines a map $\Omega : \pi_1(C_n(D^\circ), Q) \rightarrow A_n$. Concatenation concatenates crossing words; smoothing the join occurs in U , so Ω is a homomorphism.

The explicit loop σ_i has exactly one positive crossing, hence $\Omega\Phi(s_i) = s_i$. The chamber decomposition proved that every loop is homotopic to the product of the generator loops in its word, hence $\Phi\Omega$ is the identity too. We have proved

$$\boxed{A_n \cong \pi_1(C_n(D^\circ), Q).}$$

□

Corollary 5.8 (Null-homotopic words). *A word in the $s_i^{\pm 1}$ defines a null-homotopic configuration loop if and only if free insertions, cancellations, and Artin relation moves transform it into the empty word.*

Alphabeta Math

Proof. Injectivity of Φ identifies null-homotopic words with the identity of A_n . The finite relation-move criterion of Chapter 2 then gives the assertion.

□

5.6 Sources and further reading

- Joan S. Birman, *Braids, Links, and Mapping Class Groups*, Annals of Mathematics Studies 82, Princeton University Press, 1974, Chapter 1. A standard reference for geometric braids and their presentation.
- Morris W. Hirsch, *Differential Topology*, Graduate Texts in Mathematics 33, Springer, 1976, Chapters 2–3. Source for the relative smooth-approximation and parametric-transversality theorems used here; the required finite jet consequence is derived in the prerequisite section.
- Alphabeta Math, Artin Presentation Completeness and Braid Combing, and Braids as Fundamental Groups of Configuration Spaces. Directly related companion expositions; the proof above makes its own general-position input and sign conventions explicit.

6 The mapping class isomorphism

1. Configuration loops
2. Local point-motion sections
3. Lifted disk motions
4. Inverse endpoint class
5. Alexander contraction
6. Isomorphism

A collision-free motion of marked points can be extended to a motion of the whole disk. Its final homeomorphism preserves the marked set. We prove that its inverse determines a mapping class independent of every choice, and that this construction gives an isomorphism.

Theorem 6.1 (The mapping class isomorphism). *For $n \geq 3$, the map constructed below is an isomorphism*

$$\Psi : \pi_1(C_n(D^\circ), Q) \longrightarrow \text{Mod}(D, Q; \partial D).$$

If F_t is a boundary-fixed disk motion with $F_0 = \text{id}$ and $F_t(Q) = \alpha(t)$, then $\Psi([\alpha]) = [F_1^{-1}]$.
Alphabeta Math

Definition 6.2 (Homeomorphism groups and their topology). Recall $D = \{z \in \mathbb{C} : |z| \leq 1\}$, $q_j = -1 + 2j/(n+1)$, and $Q = \{q_1, \dots, q_n\}$. Let

$$\mathcal{H} = \text{Homeo}^+(D; \partial D), \quad \mathcal{K} = \{f \in \mathcal{H} : f(Q) = Q\}.$$

Both groups carry the uniform topology on maps, equivalent here to the compact-open topology. Their multiplication is ordinary composition. As in chapter 4, $\text{Mod}(D, Q; \partial D) = \pi_0(\mathcal{K})$, where π_0 denotes *path* components. A path in $\mathcal{H}$ may move the marked points; a path in $\mathcal{K}$ preserves their set at every time.

Alphabeta Math

6.1 Local extension of point motions

Lemma 6.3 (Continuous local point-motion sections). *For every $S_0 \in C_n(D^\circ)$, there is an open neighbourhood U and a continuous map $S \mapsto g_S \in \mathcal{H}$ such that $g_S(S_0) = S$ and $g_{S_0} = \text{id}$. The maps have support in disjoint interior disks.*

Alphabeta Math

Proof. Fix a configuration $S_0 = \{p_1, \dots, p_n\}$ in the open disk. Choose $r > 0$ so that the closed radius- r disks about the p_i are disjoint and contained in D° . A sufficiently nearby unordered configuration S has exactly one point s_i in each radius- $r/4$ disk. Write $v_i = s_i - p_i$. Let U be the open neighbourhood consisting of these configurations. The disjoint-disk covering chart from chapter 3 identifies U homeomorphically with the product of the radius- $r/4$ disks, so each coordinate $S \mapsto s_i$ is continuous.

On the plane define the bump and point-moving map

$$\rho_i(z) = \max\{0, 1 - |z - p_i|/r\}, \quad h_{i,S}(z) = z + \rho_i(z)v_i.$$

Then $h_{i,S}(p_i) = s_i$, and $h_{i,S}$ is the identity outside its radius- r disk. It is a homeomorphism: its perturbation of the identity has Lipschitz constant $|v_i|/r < 1/4$, so

$$|h_{i,S}(z) - h_{i,S}(w)| \geq (1 - |v_i|/r)|z - w|.$$

For every y , the equation $z = y - \rho_i(z)v_i$ has a unique solution by the contraction principle. Thus the map is onto as well as one-to-one, and its inverse is Lipschitz. Replacing v_i by uv_i , $0 \leq u \leq 1$, gives a path of homeomorphisms from the identity, so it preserves orientation. Since the complement of the supporting disk is fixed and the map is injective and onto, that disk maps to itself: an inside point cannot map to a fixed outside point, and surjectivity gives the converse inclusion.

Restricting the product to D gives

$$g_S = h_{1,S} \circ \cdots \circ h_{n,S} \in \mathcal{H}, \quad g_S(S_0) = S, \quad g_{S_0} = \text{id}.$$

The supports are disjoint, so the product sends every p_i to s_i . Moreover, $\|h_{i,S} - h_{i,T}\|_\infty \leq |s_i - t_i|$, where t_i is the point of T in the same small disk. Continuity of composition (chapter 4) therefore proves that $S \mapsto g_S$ is continuous in the uniform topology. This is a *continuous local point-motion section*. See the corresponding Alhabeta Math lemma. $\square$

Prerequisite: the contraction principle

Theorem 6.A (Contraction principle). *A contraction with constant $c < 1$ on a nonempty complete metric space has a unique fixed point.*

Alhabeta Math

Proof. If T maps a complete metric space to itself with $d(Tx, Ty) \leq cd(x, y)$, $c < 1$, the iterates $x_{m+1} = T(x_m)$ satisfy $d(x_{m+1}, x_m) \leq c^m d(x_1, x_0)$. The geometric-series estimate makes them Cauchy. Their limit is a fixed point by continuity; any two fixed points have distance at most c times their distance, so coincide. Apply this to the complete plane and $T(z) = y - \rho_i(z)v_i$.

The inverse estimate above gives continuous dependence on the target. Continuous dependence on S also follows from the explicit maps and continuity of inversion in $\mathcal{H}$, proved in chapter 4. $\square$

6.2 Global extension and path connectedness

Lemma 6.4 (Extension of a continuous point motion). *Every path $\alpha : I \rightarrow C_n(D^\circ)$ with $\alpha(0) = Q$ admits a path $F_t \in \mathcal{H}$ with $F_0 = \text{id}$ and $F_t(Q) = \alpha(t)$.*

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Proof. A continuous configuration path α , starting at Q , lifts to labeled paths $z_i(t)$ by the covering-space construction in chapter 3. Compactness gives a positive lower bound for the distances between points and from points to ∂D . Uniform continuity allows a finite subdivision of time on which each point moves only a small fraction of this bound.

On each subinterval use the local maps from step 1, centered at its initial configuration. If F_{t_j} has already been constructed, set $F_t = g_{\alpha(t)} \circ F_{t_j}$ there. Since the local section equals the identity at t_j , the pieces join continuously. Hence

$$F_0 = \text{id}, \quad F_t \in \mathcal{H}, \quad F_t(Q) = \alpha(t).$$

$\square$

Lemma 6.5 (Path connectedness). *The space $C_n(D^\circ)$ is path connected. Every configuration is $f(Q)$ for some $f \in \mathcal{H}$.*

Alphabeta Math: configurations · transitivity

Proof. The space $C_n(D^\circ)$ is path connected. One explicit construction first moves the labeled points one at a time to a temporary set disjoint from both the initial and target configurations, then to their targets. At each step a polygonal path in the disk avoiding the finitely many stationary points exists. For endpoints a, b outside a finite forbidden set, choose an intermediate point w in the open disk outside the finitely many lines through a or b and a forbidden point. Then the segments $a \rightarrow w \rightarrow b$ avoid the forbidden set and remain in the open disk by convexity. Thus every configuration is $f(Q)$ for some $f \in \mathcal{H}$. $\square$

6.3 The evaluation bundle

Definition 6.6 (Evaluation on the marked set). Define

$$e : \mathcal{H} \longrightarrow C_n(D^\circ), \quad e(f) = f(Q).$$

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Proposition 6.7 (Local triviality of evaluation). *The map $e : \mathcal{H} \rightarrow C_n(D^\circ)$ is a continuous surjective fiber bundle with fiber $\mathcal{K}$.*

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Proof. It is continuous because $f \mapsto (f(q_1), \dots, f(q_n))$ is continuous in the uniform topology and the quotient map to unordered configurations is continuous. Step 2 proves that it is onto; its fiber at Q is $e^{-1}(Q) = \mathcal{K}$. To describe it near S_0 , choose $f_0(Q) = S_0$ and put $s(S) = g_S \circ f_0$. Then

$$U \times \mathcal{K} \longrightarrow e^{-1}(U), \quad (S, k) \longmapsto s(S) \circ k$$

is a homeomorphism, with inverse $f \mapsto (e(f), s(e(f))^{-1} \circ f)$. These formulas are continuous because composition and inversion in $\mathcal{H}$ are continuous. Consequently e is a locally trivial fiber bundle with fiber $\mathcal{K}$. $\square$

The local product structure also supplies homotopy lifting. The exact version needed below is proved in the following section. This agrees with the evaluation-bundle lemma; no long exact sequence is required for our proof.

Prerequisite: fiber bundles and lifting a square

Definition 6.B (Locally trivial fiber bundle). A *locally trivial fiber bundle* $p : E \rightarrow B$ has a homeomorphism $p^{-1}(U) \cong U \times F$ over each suitable open $U \subseteq B$, taking p to projection. In one chart a lift of a path is obtained by keeping its fiber coordinate constant.

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Theorem 6.C (Lifting a square). *We also need this statement: a map $H : I \times I \rightarrow B$, with a chosen lift along the bottom edge, has a lift on the whole square. Chosen lifts on the vertical edges can be preserved when they agree at the bottom corners. Here $I = [0, 1]$.*

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Proof. Pull back the trivializing cover to the compact square. A Lebesgue-number subdivision yields finitely many rectangles, each mapping into one chart. Work row by row and within each row from left to right. On each new rectangle the lift is already specified on its bottom edge, usually also its left edge, and sometimes its right edge. In a product chart the base coordinate is given by H . Extend the fiber coordinate by composing its existing edge values with a retraction of the rectangle onto those edges. A rectangle retracts onto its bottom edge, onto its bottom and left edges, and onto its bottom and both vertical edges. For the L-shaped union in a unit square, use

$$(x, y) \mapsto (\max(x - y, 0), \max(y - x, 0)).$$

For the U-shaped union, continue the ray from $(1/2, 2)$ through each point until it hits a side or the bottom. Explicitly, put

$$\lambda(x, y) = \frac{1}{\max\{1 - y/2, 2|x - 1/2|\}}, \quad R(x, y) = (1/2 + \lambda(x, y)(x - 1/2), 2 + \lambda(x, y)(y - 2)).$$

The denominator is at least $1/2$. The image lies on the bottom or a vertical side, and $\lambda = 1$ on those edges. Thus this formula is a continuous retraction. Rescaling gives these retractions on every grid rectangle. The resulting lifts agree on shared edges, so the finite pasting lemma gives a continuous lift. This argument includes the relative edge conditions just stated.

This is the square case of Hatcher, *Algebraic Topology*, Proposition 4.48, pp. 379–380. We use this theorem for bundles, whose lifts need not be unique; covering maps have discrete fibers and unique based path lifts.

□

6.4 The inverse-endpoint homomorphism

Proposition 6.8 (Well-definedness and multiplication). *If $F_0 = \text{id}$ and $F_t(Q) = \alpha(t)$, the rule*

$$\Psi([\alpha]) = [F_1^{-1}].$$

defines a homomorphism $\Psi : \pi_1(C_n(D^\circ), Q) \rightarrow \text{Mod}(D, Q; \partial D)$, independent of the lift and the loop representative.

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Proof. Let α be a loop at Q , and choose a lift F_t in $\mathcal{H}$ starting at the identity. Since $F_1(Q) = Q$, its inverse lies in $\mathcal{K}$. Define

Choice of lift. For another such lift G_t , the path $k_t = F_t^{-1} \circ G_t$ lies in $\mathcal{K}$ and starts at the identity. Thus $[k_1] = 1$, and $G_1 = F_1 \circ k_1$ gives $[G_1^{-1}] = [F_1^{-1}]$.

Choice of loop representative. An endpoint-fixed homotopy $H(s, t)$ between loops lifts to $\tilde{H}(s, t)$ with $\tilde{H}(s, 0) = \text{id}$, by the square-lifting result. At $t = 1$, the path $s \mapsto \tilde{H}(s, 1)$ lies in $\mathcal{K}$. Its inverse path shows that the two endpoint inverse classes agree. The choice-of-lift argument compares these lifts with any previously chosen ones.

Multiplication. Our product $\alpha * \beta$ traverses α first. If F_t, G_t lift the two loops, a lift of their product first follows F_t , then $G_t \circ F_1$. Its endpoint is $G_1 \circ F_1$, so

$$\Psi([\alpha][\beta]) = [(G_1 \circ F_1)^{-1}] = [F_1^{-1} \circ G_1^{-1}] = \Psi([\alpha])\Psi([\beta]).$$

□

The inverse is required by our convention. Sending a motion to $[F_1]$ would reverse the order of multiplication. Compare the point-motion homomorphism lemma.

6.5 The Alexander contraction

Theorem 6.9 (Contractibility of the unmarked homeomorphism group). *For $f \in \mathcal{H}$ and $0 < u \leq 1$, define the family below and set $A_0(f) = \text{id}$. This family is jointly continuous in u and f , consists of boundary-fixed orientation-preserving homeomorphisms, and satisfies $A_0(f) = \text{id}$, $A_1(f) = f$, and $A_u(\text{id}) = \text{id}$. Hence $\mathcal{H}$ is contractible.*

$$A_u(f)(z) = \begin{cases} u f(z/u), & |z| \leq u, \\ z, & |z| \geq u, \end{cases} \quad A_0(f) = \text{id}.$$

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Proof. The formulas agree at $|z| = u$, since f fixes the unit circle. The inverse is $A_u(f^{-1})$, so every $A_u(f)$ is a boundary-fixed homeomorphism. It preserves orientation: on the inner disk it is a conjugate of f , and outside it is the identity. We have $A_1(f) = f$ and $A_u(\text{id}) = \text{id}$.

The estimate $\|A_u(f) - \text{id}\|_\infty \leq 2u$ proves continuity at $u = 0$, uniformly in f . At positive u , the glued formulas are jointly continuous in (u, z) for fixed f ; compactness gives continuity in the uniform norm. Finally,

$$\|A_u(f) - A_u(g)\|_\infty \leq \|f - g\|_\infty$$

gives joint continuity in (u, f) . Thus A_u contracts $\mathcal{H}$ to the identity. This is the Alexander contraction. □

This contraction need not preserve Q . It contracts the *unmarked* group $\mathcal{H}$, which is precisely what the argument needs.

6.6 Proof of the mapping class isomorphism

Proof of Theorem 6.1. Surjectivity. Given a mapping class $[k]$, $k \in \mathcal{K}$, take the path $F_t = A_t(k^{-1})$ from the identity to k^{-1} . Its projected configuration path $\alpha(t) = F_t(Q)$ is a loop. By construction $\Psi([\alpha]) = [k]$.

Injectivity. Suppose $\Psi([\alpha]) = 1$. For a chosen lift F_t , this means F_1 is joined to the identity by a path k_t in $\mathcal{K}$. Concatenate F_t with k_t to obtain a based loop L in $\mathcal{H}$. Its evaluation is α followed by a constant loop, so represents $[\alpha]$. The based contraction

$$(u, t) \mapsto A_u(L(t))$$

deforms L to the constant loop while keeping its endpoints the identity. Evaluating at Q gives an endpoint-fixed null homotopy of α . Hence $[\alpha] = 1$, proving injectivity.

Together with the homomorphism calculation, this proves the theorem. See also the evaluation-bundle isomorphism theorem. □

6.7 The three models and pure subgroups

Corollary 6.10 (Comparison of braid groups and pure subgroups).

$$A_n \xrightarrow{\cong} \pi_1(C_n(D^\circ), Q) \xrightarrow{\Psi} \text{Mod}(D, Q; \partial D).$$

The isomorphism sends s_i to the clockwise half-twist class and restricts to $P_n \cong \text{PMod}(D, Q; \partial D)$.

Alphabeta Math: braid models · pure subgroups

Proof. A counterclockwise elementary exchange lifts to a counterclockwise half rotation supported near two adjacent marked points. The inverse endpoint is the clockwise half twist used in chapter 4. Combining Ψ with chapter 5 therefore identifies the Artin generators with those clockwise mapping classes:

The construction restricts to an isomorphism of pure subgroups. Indeed, $t \mapsto (F_t(q_1), \dots, F_t(q_n))$ is the unique ordered lift of the loop starting at the base tuple. If its forward endpoint permutation is e_α , then $F_1(q_j) = q_{e_\alpha(j)}$; the puncture permutation of F_1^{-1} is e_α^{-1} . Thus Ψ intertwines exactly the permutation homomorphisms defined in chapters 3 and 4. Their kernels correspond in both directions under the isomorphism, giving $P_n \cong \text{PMod}(D, Q; \partial D)$. □

6.8 Sources

The linked Alfabeta Math items supply the local-section, lifting, contraction, and isomorphism statements. The standard braid interpretation is discussed in González-Meneses, *Basic results on braid groups*, §1.4. For bundle lifting and its general homotopy-theoretic form, see Hatcher, *Algebraic Topology*, Proposition 4.48 and Theorem 4.41. Our argument proves bijectivity directly instead of assuming the long exact sequence.

References

- [1] Juan González-Meneses, *Basic results on braid groups*, Annales mathématiques Blaise Pascal **18** (2011), no. 1, 15–59. doi:10.5802/ambp.293; arXiv:1010.0321. Sections 1.2–1.4 discuss the classical braid models.
- [2] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, 2002. Author’s freely available text. Chapter 1, §1.3, Propositions 1.30–1.31, and Chapter 4, Proposition 4.48 and Theorem 4.41, supply covering and bundle lifting results.
- [3] *Alphabeta Math*, alphabeta math .cc. Individual definitions, lemmas, and theorem sources are linked at their statements throughout these notes.
- [4] Get Curious, *Braid groups*, six-chapter article series, October 2026. Series overview. This PDF retains the series’ statement numbering and conventions.